2.18 Slope $\frac{\partial z}{\partial x}$ of $z = 4x^2 - xy + y^2 - y$ graph $\cap \{y=1\}$ at $(-3, 1, 39)$

$$\frac{\partial z}{\partial x} = 8x - y = -24 - 1 = \boxed{-25}$$

Meanings: Derive: prove "Derive equation 1 from equation 2."

Differentiate: take the derivative.

Warm-ups Let $F(x,y,z) = y \arcsin(y\sqrt{x})$

1. Find F_y
2. Find F_x
3. Find $\frac{\partial F}{\partial u}$, where u = unit vector in direction $(3,-4)$ at $(x,y) = (1, \frac{1}{2})$.

1. Prod rule

$$F_y = \arcsin(y\sqrt{x}) + y \frac{1}{\sqrt{1-(y\sqrt{x})^2}} \cdot \sqrt{x}$$

$$F_y = \arcsin(y\sqrt{x}) + \frac{y\sqrt{x}}{\sqrt{1-y^2x}}.$$

2. chain

$$F_x = y \frac{1}{\sqrt{1-(y\sqrt{x})^2}} \cdot y \cdot \frac{1}{2}x^{-1/2} = \frac{y^2}{2\sqrt{x}\sqrt{1-y^2x}}$$

3. $\frac{\partial F}{\partial u}$ u = unit vector in direction $(3,-4) \Rightarrow u = \left(\frac{3}{\sqrt{9+16}}, \frac{-4}{\sqrt{9+16}}\right)$

at $(x,y) = (1, \frac{1}{2})$

$$u = \left(\frac{3}{5}, \frac{-4}{5}\right)$$

$$\Rightarrow \frac{\partial F}{\partial u} = F_x\left(\frac{3}{5}\right) + F_y\left(\frac{-4}{5}\right).$$

$$F_y = \arcsin\left(\frac{1}{2}\right) + \frac{\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}$$

$$= \frac{\pi}{6} + \frac{\frac{1}{2}}{\sqrt{\frac{3}{4}}} = \frac{\pi}{6} + \frac{1}{\sqrt{3}}.$$

$$F_x = \frac{\frac{1}{4}}{2\cdot 1\sqrt{3/4}} = \frac{1}{8\cdot\frac{\sqrt{3}}{2}} = \frac{1}{4\sqrt{3}}.$$

$$\frac{\partial F}{\partial u} = F_x\left(\frac{3}{5}\right) + F_y\left(\frac{-4}{5}\right) = \frac{1}{4\sqrt{3}}\cdot\frac{3}{5} + \left(\frac{\pi}{6}+\frac{1}{\sqrt{3}}\right)\left(-\frac{4}{5}\right)$$

$$= \boxed{\frac{\sqrt{3}}{20} + \frac{-2\pi}{15} - \frac{4}{5\sqrt{3}}}.$$

Question: If it's a cold day, and $T(x,y,z)$ is the temperature at position (x,y,z) in our classroom, for which unit vector u at Dr. Richardson's iPad would yield the greatest $\frac{\partial T}{\partial u}$?

(Ans: unit vector pointing at heat vent closest.)

Lot's of other interesting derivatives:

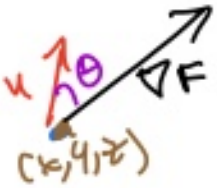
Observe that if $F(x,y,z\cdots)$ is a function, $u=(u_1,u_2,u_3,\cdots)$ is a unit vector, then

$$\frac{\partial F}{\partial u} = F_x u_1 + F_y u_2 + F_z u_3 + \cdots$$

$$= \underbrace{(F_x, F_y, F_z \cdots)}_{\text{"go frogs of F"}} \cdot (u_1, u_2, u_3 \cdots)$$

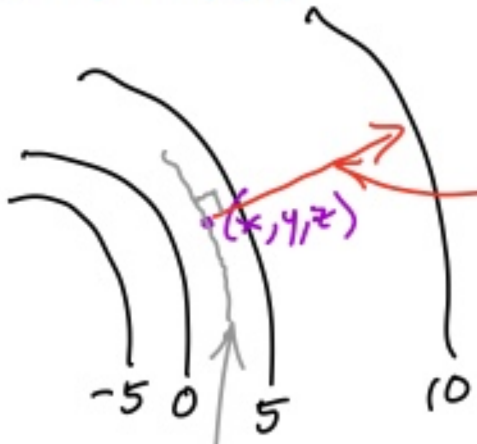
$$\underbrace{\nabla F}_{\text{"gradient of F"}} = (F_x, F_y, F_z, \ldots) \leftarrow \text{a vector!}$$

$$\Rightarrow \frac{\partial F}{\partial u} = \nabla F \cdot u = |\nabla F|\,|u| \cos\theta$$
$$= |\nabla F| \cos\theta$$



Notice:
- $\frac{\partial F}{\partial u} = 0$ when $u \perp \nabla F$ $\quad \theta = \frac{\pi}{2}$, $\theta = -\frac{\pi}{2}$
- $\frac{\partial F}{\partial u} =$ maximum when $u = \frac{\nabla F}{|\nabla F|}$ (unit vector in same direction as ∇F) $\quad \theta = 0$
- $\frac{\partial F}{\partial u} = \left(\text{largest negative \#}\right)$ when $u = \frac{-\nabla F}{|\nabla F|}$ $\quad \theta = \pi$

Given a pt. (x,y,z)

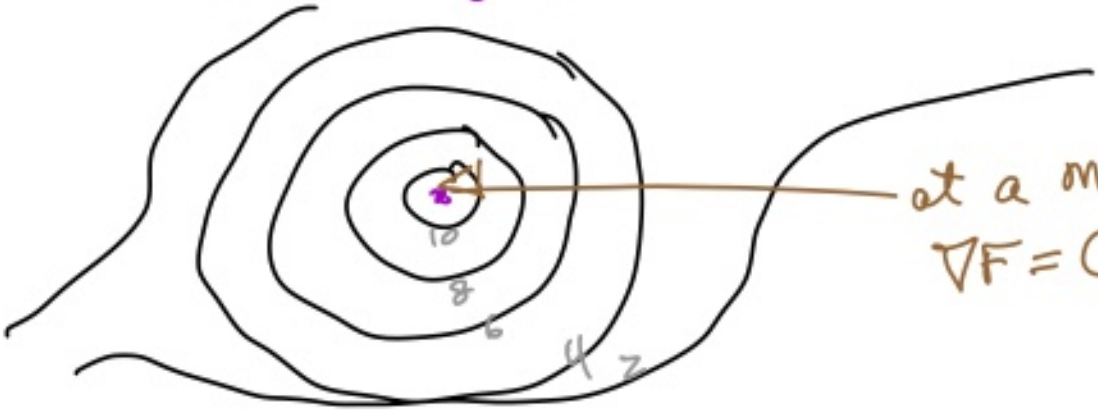


∇F points in the direction of greatest increase.

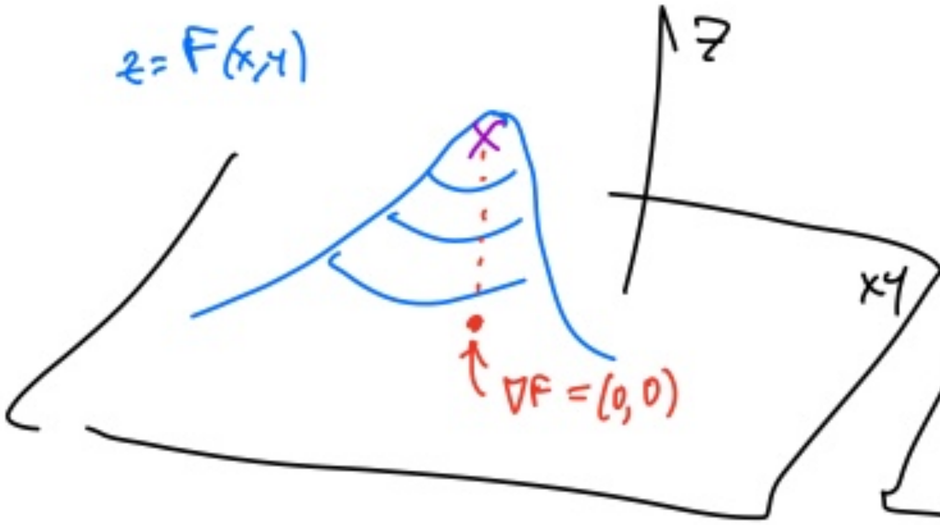
$|\nabla F|$ = instantaneous rate of change in that direction.

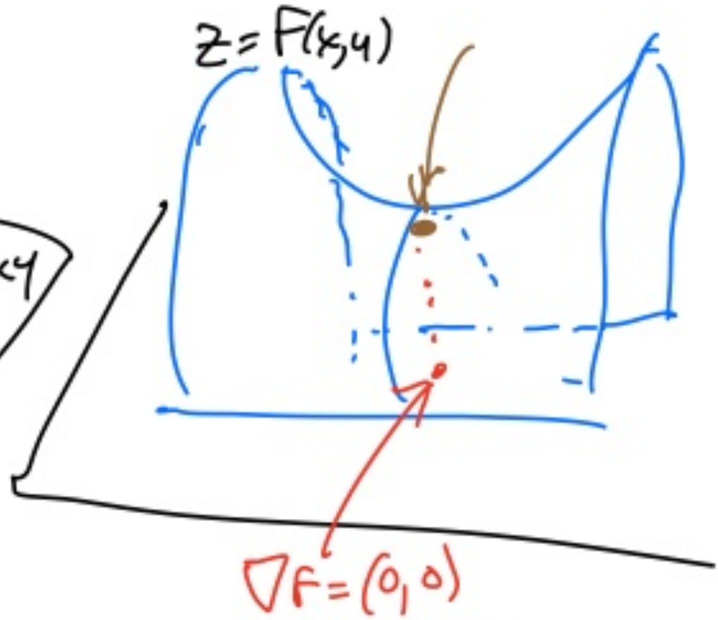
∇F is $\perp$ to level set (contour) through that point.

Another use of ∇F:



at a max or min, $\nabla F = 0 = (0, 0, \dots)$





$\nabla F = (0, 0, \ldots)$

at (x,y) = local min, local max, or a saddle pt.

A critical point is a pt where $\nabla F = (0,0,\ldots)$ or ∇F is undefined.

Related to the gradient: the differential dF

(used with integrals)

$$df(x) = f'(x)\,dx$$

$$d(g(x,y)) = g_x(x,y)\,dx + g_y(x,y)\,dy$$

$$d(h(x,y,z)) = h_x\,dx + h_y\,dy + h_z\,dz$$

$$= \nabla h \cdot (dx, dy, dz)$$

Example: $G(x,y,z) = \sqrt{z+xy}$

$$dG = \tfrac{1}{2}(z+xy)^{-1/2}\cdot y\,dx + \tfrac{1}{2}(z+xy)^{-1/2}\cdot x\,dy + \tfrac{1}{2}(z+xy)^{-1/2}dz$$

$$\nabla G = \left(\tfrac{1}{2}(z+xy)^{-1/2}\cdot y,\ \tfrac{1}{2}(z+xy)^{-1/2}\cdot x,\ \tfrac{1}{2}(z+xy)^{-1/2}\right)$$

$$= \frac{1}{2\sqrt{z+xy}}(y, x, 1).$$

Last new derivative : derivative matrix

$$F : \underbrace{\mathbb{R}^n}_{\text{domain}} \to \underbrace{\mathbb{R}^k}_{\text{codomain}}$$

$$F(x_1, x_2, \dots, x_n) = \begin{pmatrix} F_1(x_1, x_2, \dots, x_n) \\ F_2(x_1, x_2, \dots, x_n) \\ \vdots \\ F_k(x_1, x_2, \dots, x_n) \end{pmatrix}$$

$$F'(x_1, x_2, \dots, x_n) = \begin{pmatrix} - \nabla F_1 - \\ - \nabla F_2 - \\ \vdots \\ - \nabla F_k - \end{pmatrix}$$

derivative matrix.

rows columns

$n \times k$ matrix

Example : $F(x,y,z) = (\underbrace{2x^2y}_{F_1}, \underbrace{z^2 + yxz}_{F_2})$

$F : \mathbb{R}^3 \to \mathbb{R}^2$

$$F' = \begin{pmatrix} \nabla F_1 \\ \nabla F_2 \end{pmatrix} = \begin{pmatrix} 4xy & 2x^2 & 0 \\ yz & xz & (2z+yx) \end{pmatrix}$$

Chain Rule for the Derivative matrix :

$F : \mathbb{R}^n \to \mathbb{R}^k, \quad G : \mathbb{R}^k \to \mathbb{R}^p$

$$\mathbb{R}^n \xrightarrow{F} \mathbb{R}^k \xrightarrow{G} \mathbb{R}^p$$

$G \circ F : \mathbb{R}^n \to \mathbb{R}^p$

rows columns

G' $k \times p$ matrix

F' $n \times k$ matrix

$$(G\circ F)'(\underbrace{x_1, x_2, \dots, x_n}_{x}) = (G\circ F)'(x)$$

$$(G(F(x)))' = \underset{n\times k}{G'(F(x))} \cdot \underset{k\times p}{F'(x)} = (n\times p) \text{ matrix}$$

same formula is true in higher dimensions

but the multiplication is <u>matrix multiplication</u>

<u>Matrix Math</u>.

Addition & subtraction.

$$\begin{pmatrix} 2 & 3 \\ 4 & 8 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$$

matrices must have same # rows x # columns

$$= \begin{pmatrix} 2+(-1)-0 & 3+0-0 \\ 4+1-2 & 8+1-2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 3 & 7 \end{pmatrix}$$

Scalar multiplication

$$7\begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 14 & -7 \\ 7 & 28 \end{pmatrix}$$

<u>matrix mult</u>.

row x # columns

$$\underset{n\times k}{A}\ \underset{k\times p}{B} \leftarrow \text{an } n\times p \text{ matrix}$$

must match

each entry is a dot product (row of A)$\cdot$(column of B)

Examples

$$\begin{array}{cc} A & B \\ \begin{pmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 3 & 1 & 4 \\ 6 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix} \end{array}$$

$2 \times 3 \quad 3 \times 3$

ans 2×3

$$= \begin{pmatrix} \begin{pmatrix} 1^{st} \\ \text{row} \end{pmatrix}_A \cdot \begin{pmatrix} 1^{st} \\ \text{column} \end{pmatrix}_B & \begin{pmatrix} 1^{st} \\ \text{row} \end{pmatrix} \cdot \begin{pmatrix} 2^{nd} \\ \text{col} \end{pmatrix} & \begin{pmatrix} 1^{st} \\ \text{row} \end{pmatrix} \cdot \begin{pmatrix} 3^{rd} \\ \text{colm} \end{pmatrix} \\ \begin{pmatrix} 2^{nd} \\ \text{row} \end{pmatrix} \cdot \begin{pmatrix} 1^{st} \\ \text{column} \end{pmatrix} & \begin{pmatrix} 2^{nd} \\ \text{row} \end{pmatrix} \cdot \begin{pmatrix} 2^{nd} \\ \text{col} \end{pmatrix} & \begin{pmatrix} 2^{nd} \\ \text{row} \end{pmatrix} \cdot \begin{pmatrix} 3^{rd} \\ \text{col} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (2,1,0)\cdot(3,6,0) & (2,1,0)\cdot(1,1,2) & (2,1,0)\cdot(4,0,2) \\ (-1,1,1)\cdot(3,6,0) & (-1,1,1)\cdot(1,1,2) & (-1,1,1)\cdot(4,0,2) \end{pmatrix}$$

$$= \begin{pmatrix} 12 & 3 & 8 \\ 3 & 2 & -2 \end{pmatrix}$$

Practice

$$\begin{pmatrix} 3 & -1 \\ 2 & 4 \\ -1 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$3 \times \underline{\underline{2}} \quad \underline{\underline{2}} \times 4$

$$\begin{pmatrix} \underline{3} & \underline{-1} & \underline{6} & \underline{2} \\ \underline{2} & \underline{4} & \underline{4} & \underline{6} \\ \underline{1} & \underline{8} & \underline{2} & \underline{9} \end{pmatrix}$$

<u>Note</u>: Matrix multiplication is <u>not</u> commutative.